ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
Vol. 7, Issue 4, pp: (38-42), Month: October - December 2019, Available at: www.researchpublish.com

Partial Differential Problem of Two Types of Complicated Trigonometric Functions

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong Province, China

Abstract: **In this paper, we study the partial differential problem of two types of two variables trigonometric functions. The infinite series expressions of any order partial derivatives of the two types of two variables functions can be obtained by differentiation term by term theorem. Therefore, the difficulty of evaluating the higher order partial derivatives of these two variables trigonometric functions can be greatly reduced. Moreover, we provide some examples to do calculation practically. The research method adopted is to find solutions through manual calculations, and verify these solutions using Maple. This research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking. Therefore, Maple provides insights and guidance regarding problem-solving methods.**

Keywords: **Partial differential problem, Trigonometric functions, Infinite series expressions, Differentiation term by term theorem, Maple.**

I. INTRODUCTION

This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus, engineering mathematics and physics, the study of partial differential problem of multivariable functions is an important issue. For example, Laplace equations, wave equations, and other important physical equations are involved the partial derivatives of multivariable functions. Therefore, whether in physics, engineering or other sciences, the evaluation and numerical calculation of partial derivatives of multivariable functions has its importance. This paper studies the partial differential problem of the following two types of two variables trigonometric functions:

$$f(r,\theta) = arctan\left(\frac{2rcos\theta}{1-r^2}\right), \tag{1}$$

and

$$g(r,\theta) = arctanh\left(\frac{2rsin\theta}{1+r^2}\right), \tag{2}$$

where r, θ are real numbers, and $|r| < 1$. By differentiation term by term theorem, we can determine the infinite series expressions of any order partial derivatives of the two types of two variables trigonometric functions. Therefore, the difficulty of evaluating the higher order partial derivatives of these two variables functions can be greatly reduced. [1-5] provided some methods to evaluate the partial derivatives of multivariable functions, which are different from the methods used in this paper. On the other hand, [6-15] used some techniques, for example, complex power series, binomial series, and differentiation term by term theorem to study the partial differential problem. In this paper, we propose two examples of two variables functions to evaluate their any order partial derivatives, and determine some of their higher order partial derivative values practically. On the other hand, we employ Maple to calculate the approximations of these higher order partial derivative values and their infinite series expressions to verify our answers.

ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
Vol. 7, Issue 4, pp: (38-42), Month: October - December 2019, Available at: www.researchpublish.com

II. PRELIMINARIES AND MAJOR RESULTS

Firstly, some properties used in this paper are introduced below.

2.1 Definitions:

2.1.1 Let m, n be non-negative integers. The $(m+n)$-th order partial derivative (m -times partial derivatives with respect to θ, and n -times partial derivatives with respect to r) of a two variables function $f(r,\theta)$, is denoted by $\frac{\partial^{m+n} f}{\partial\theta^m \partial r^n}(r,\theta)$.

2.1.2 Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, and a, b are real numbers. a , the real part of z , is denoted by $\text{Re}(z)$; b , the imaginary part of z , is denoted by $\text{Im}(z)$.

2.1.3 Assume that r is a real number and k is a positive integer. Define $(r)_k = r(r-1)\cdots(r-k+1)$, and $(r)_0 = 1$.

2.2 Formulas:

2.2.1 Euler's formula :

$exp(i\theta) = cos\theta + isin\theta$, where θ is a real number.

2.2.2 DeMoivre's formula

$(cos\theta + isin\theta)^p = cosp\theta + isinp\theta$, where p is an integer, and θ is a real number.

2.2.3 Suppose that z is a complex number, then $arctanz = \frac{1}{2i} ln\left(\frac{1+iz}{1-iz}\right)$ for $z \neq -i$, (3)

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} z^{2k+1} \text{ for } |z| < 1. \tag{4}$$

2.2.4 $arctanh = \frac{1}{2} ln\left(\frac{1+z}{1-z}\right)$ for $z \neq 1$. (5)

2.3 Differentiation term by term theorem:([16, p230])

If, for all non-negative integer k , the functions $g_k: (a,b) \to R$ satisfy the following three conditions: (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a,b), (iii) $\sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$ is uniformly convergent on (a,b). Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a,b) , and its derivative $\frac{d}{dx} \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{d}{dx} g_k(x)$.

To obtain the major results, the following lemma is needed.

Lemma 1 *Suppose that* x, y *are real numbers with* $x^2 + y^2 \neq 1$,*then*

$$arctan(x+iy) = \frac{1}{2} arctan\left(\frac{2x}{1-x^2-y^2}\right) + i\frac{1}{2} arctanh\left(\frac{2y}{1+x^2+y^2}\right). \tag{6}$$

Proof.

$$arctan(x+iy) = \frac{1}{2i} ln\left(\frac{1-y+ix}{1+y-ix}\right) \quad \text{(by Eq. (3))}$$

$$= \frac{1}{2i} ln\left(\frac{(1-x^2-y^2)+i2x}{(1+y)^2+x^2}\right)$$

$$= \frac{1}{2i} ln\left[\frac{\sqrt{(1-x^2-y^2)^2+4x^2}}{(1+y)^2+x^2} \cdot \left(\frac{1-x^2-y^2}{\sqrt{(1-x^2-y^2)^2+4x^2}} + i\frac{2x}{\sqrt{(1-x^2-y^2)^2+4x^2}}\right)\right]$$

$$= \frac{1}{2} arctan\left(\frac{2x}{1-x^2-y^2}\right) - i\frac{1}{2} ln\left(\frac{\sqrt{(1-x^2-y^2)^2+4x^2}}{(1+y)^2+x^2}\right)$$

$$= \frac{1}{2} arctan\left(\frac{2x}{1-x^2-y^2}\right) + i\frac{1}{2} arctanh\left(\frac{2y}{1+x^2+y^2}\right). \quad \text{(by Eq. (5))}$$

Q.e.d.

In the following, we obtain the infinite series expressions of any order partial derivatives of the two variables functions (1) and (2).

Theorem 1 *Assume that* m, n *are non-negative integers,* r, θ *are real numbers,* $|r| < 1$, *and let*

$$f(r,\theta) = arctan\left(\frac{2rcos\theta}{1-r^2}\right), \textit{ then}$$

$$\frac{\partial^{m+n} f}{\partial\theta^m \partial r^n}(r,\theta) = 2\sum_{k=0}^{\infty}(-1)^k (2k+1)_n (2k+1)^{m-1} r^{2k+1-n} cos\left[(2k+1)\theta + \frac{m\pi}{2}\right]. \tag{7}$$

Proof. In Eq. (4), let $z = re^{i\theta}$, by Euler's formula and DeMoivre's formula, we have

$$arctan\left(re^{i\theta}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} r^{2k+1} e^{i(2k+1)\theta}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} r^{2k+1} [cos(2k+1)\,\theta + isin(2k+1)\theta] \,. \tag{8}$$

On the other hand, using Lemma 1 yields

$$arctan\left(re^{i\theta}\right) = \frac{1}{2} arctan\left(\frac{2rcos\theta}{1-r^2}\right) + i\frac{1}{2} arctanh\left(\frac{2rsin\theta}{1+r^2}\right). \tag{9}$$

Hence, by the equality of real parts of both sides of Eq. (9), we obtain

$$f(r,\theta) = 2\mathrm{Re}\left[arctan\left(re^{i\theta}\right)\right]$$

$$= 2\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} r^{2k+1} cos[(2k+1)\,\theta]. \tag{10}$$

In Eq.(10), differentiating *m*-times with respect to θ , and *n*-times with respect to *r*, and using differentiation term by term theorem yields

$$\frac{\partial^{m+n} f}{\partial\theta^m \partial r^n}(r,\theta) = 2\sum_{k=0}^{\infty}(-1)^k (2k+1)_n (2k+1)^{m-1} r^{2k+1-n} cos\left[(2k+1)\theta + \frac{m\pi}{2}\right].$$

Q.e.d.

Theorem 2 *Suppose that the assumptions are the same as Theorem 1, and let*

$$g(r,\theta) = arctanh\left(\frac{2rsin\theta}{1+r^2}\right), \textit{ then}$$

$$\frac{\partial^{m+n} g}{\partial\theta^m \partial r^n}(r,\theta) = 2\sum_{k=0}^{\infty}(-1)^k (2k+1)_n (2k+1)^{m-1} r^{2k+1-n} sin\left[(2k+1)\theta + \frac{m\pi}{2}\right]. \tag{11}$$

Proof. By the equality of imaginary parts of both sides of Eq. (9), we have

$$g(r,\theta) = 2\mathrm{Im}\left[arctan\left(re^{i\theta}\right)\right]$$

$$= 2\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} r^{2k+1} sin[(2k+1)\,\theta]. \tag{12}$$

In Eq.(12), differentiating *m*-times with respect to θ , and *n*-times with respect to *r*, , then by differentiation term by term theorem, we obtain

$$\frac{\partial^{m+n} g}{\partial\theta^m \partial r^n}(r,\theta) = 2\sum_{k=0}^{\infty}(-1)^k (2k+1)_n (2k+1)^{m-1} r^{2k+1-n} sin\left[(2k+1)\theta + \frac{m\pi}{2}\right].$$

Q.e.d.

III. EXAMPLES

In the following, for the partial differential problem of two types of two variables trigonometric functions discussed in this study, we provide two examples and use Theorems 1 and 2 to determine the infinite series expressions of their any order partial derivatives. Moreover, we employ Maple to calculate the approximations of some of their higher order derivatives values for verifying our answers.

Example 1. In Theorem 1, let $m = 3, n = 5, r = \frac{1}{2}, \theta = \frac{\pi}{3}$, then

$$\frac{\partial^8 f}{\partial\theta^3 \partial r^5}\left(\frac{1}{2},\frac{\pi}{3}\right) = 2\sum_{k=0}^{\infty}(-1)^k (2k+1)_5 (2k+1)^2 \left(\frac{1}{2}\right)^{2k-4} cos\left[\frac{(2k+1)\pi}{3}+\frac{3\pi}{2}\right]. \tag{13}$$

We employ Maple to calculate the approximations of both sides of Eq. (13) as follows:

>f:=(r,theta)->arctan(2*r*cos(theta)/(1-r^2));

$$f := (r,\theta) \to \arctan\left(\frac{2\,r\cos(\theta)}{1-r^2}\right)$$

>evalf(D[1$5,2$3](f)(1/2,Pi/3),18);

1.74527918961272355*10^5

>evalf(2*sum((-1)^k*product(2*k+1-j,j=0..4)*(2*k+1)^2*(1/2)^(2*k-4)*cos((2*k+1)*Pi/3+3*Pi/2),k=0..infinity),18);

1.74527918961272355*10^5

Example 2. In Theorem 2, if $m = 3, n = 4, r = \frac{1}{5}, \theta = \frac{\pi}{9}$, then

$$\frac{\partial^7 g}{\partial\theta^3 \partial r^4}\left(\frac{1}{5},\frac{\pi}{9}\right) = 2\sum_{k=0}^{\infty}(-1)^k (2k+1)_4 (2k+1)^2 \left(\frac{1}{5}\right)^{2k-3} sin\left[\frac{(2k+1)\pi}{9}+\frac{3\pi}{2}\right]. \tag{14}$$

We also use Maple to find the approximations of both sides of Eq. (14).

>g:=(r,theta)->arctanh(2*r*sin(theta)/(1+r^2));

$$g := (r,\theta) \to \operatorname{arctanh}\left(\frac{2\,r\sin(\theta)}{1+r^2}\right)$$

>evalf(D[1$4,2$3](g)(1/5,Pi/9),18);

−157.494418163628719

>evalf(2*sum((-1)^k*product(2*k+1-j,j=0..3)*(2*k+1)^2*(1/5)^(2*k-3)*sin((2*k+1)*Pi/9+3*Pi/2),k=0..infinity),18);

−157.494418163628729

IV. CONCLUSION

In this article, we mainly use differentiation term by term theorem to solve the partial differential problem of two variables trigonometric functions. In fact, the applications of differentiation term by term theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and use Maple to verify our answers. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

REFERENCES

[1] C. H., Bischof, G. Corliss, and A. Griewank,“ Structured second and higher-order derivatives through univariate Taylor series,” Optimization Methods and Software, vol. 2, pp. 211-232, 1993.

[2] D. N. Richard, “ An efficient method for the numerical evaluation of partial derivatives of arbitrary order, ” ACM Transactions on Mathematical Software, vol. 18, no. 2, pp. 159-173, 1992.

[3] A. Griewank and A. Walther, Evaluating derivatives: principles and techniques of algorithmic differentiation, 2nd ed., Philadelphia: SIAM, 2008.

[4] L. E. Fraenkel, “ Formulae for high derivatives of composite functions,” Mathematical Proceedings of the Cambridge Philosophical Society, vol. 83, pp.159-165, 1978.

[5] T-W, Ma, “ Higher chain formula proved by combinatorics,” The Electronic Journal of Combinatorics , vol. 16, #N21, 2009.

ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
Vol. 7, Issue 4, pp: (38-42), Month: October - December 2019, Available at: www.researchpublish.com

[6] C. -H. Yu, " Partial derivatives of some types of two-variables functions," Pure and Applied Mathematics Journal, vol. 2, no. 2, pp. 56-61, 2013.

[7] C. -H. Yu, " Using Maple to evaluate the partial derivatives of two-variables functions," International Journal of Computer Science and Mobile Computing, vol. 2, issue. 6, pp. 225-232, 2013.

[8] C. -H. Yu, " Using Maple to study the partial differential problems," Applied Mechanics and Materials, vols. 479-480(2014), pp. 800-804, 2013.

[9] C. -H. Yu, "Using differentiation term by term theorem to study the partial differential problems," Turkish Journal of Analysis and Number Theory, vol. 1, no. 1, pp. 63-68, 2013.

[10] C. -H. Yu, " Partial derivatives of three variables functions," Universal Journal of Computational Mathematics, vol. 2, no. 2, pp. 23-27, 2014.

[11] C. -H. Yu, "Application of differentiation term by term theorem on the partial differential problems," International Journal of Partial Differential Equations and Applications, vol. 2, no. 1, pp. 7-12, 2014.

[12] C. -H. Yu, " Evaluating the partial derivatives of some types of multivariable functions," American Journal of Computing Research Repository, vol. 2, no. 1, pp. 15-18, 2014.

[13] C. -H. Yu, " Solving the partial differential problems using differentiation term by term theorem," Journal of Automation and Control, vol. 2, no. 1, pp. 8-14, 2014.

[14] C. -H. Yu, " Partial differential problems of four types of two-variables functions," American Journal of Numerical Analysis, vol. 2, no. 1, pp.4-10, 2014.

[15] C. -H. Yu, " Evaluating partial derivatives of two-variables functions by using Maple," Proceedings of the 6th IEEE/International Conference on Advanced Infocomm Technology, Hsinchu, Taiwan, no. 00295, 2013.

[16] T. M. Apostol, Mathematical analysis, 2nd ed., Boston: Addison-Wesley, 1975.